THE RESIDUE THEOREM AND AN ANALOG OF P. APPELL'S FORMULA FOR FINITE RIEMANN SURFACES

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Abstract: A theory of multiplicative functions and Prym differentials for the case of special characters on compact Riemann surfaces has found applications in geometrical function theory of complex variable, analytical number theory and in equations of mathematical physics. Theory of functions on compact Riemann surfaces differs from the theory of functions on finite Riemann surfaces even for the class of single meromorphic functions and Abelian differentials. In this article we continue the construction of the general function theory on finite Riemann surfaces for multiplicative meromorphic functions and differentials. We have proved analogues of the theorem on the full sum of residues for Prym differentials of every integral order and P. Appell's formula on expansion of the multiplicative function with poles of arbitrary multiplicity in the sum of elementary Prym integrals.

Keywords: Teichmuller spaces for finite Riemann surfaces, Prym differentials, group of characters, Jacobi manifolds

INTRODUCTION

Theory of multiplicative functions and Prym differentials for the case of special characters on compact Riemann surface has found applications in geometrical function theory of complex variable, analytical number theory and in equations of mathematical physics [1–7]. In [2, 4] the construction of the general theory of multiplicative functions and Prym differentials on compact Riemann surface for arbitrary characters was started. Theory of functions on compact Riemann surfaces differs from the theory of functions on finite Riemann surfaces even for the class of single meromorphic functions and Abelian differentials. A series of general spaces of functions and differentials on finite Riemann surface F' of genus $(g, n), g \geq 1, n > 0$ will be infinite-dimensional.

In this article we continue the construction of the general function theory on finite Riemann surfaces for multiplicative meromorphic functions and differentials. Analogues of the theorem on the full sum of residues for Prym differentials of every integral order and P. Appell's formula on expansion of the multiplicative function with poles of arbitrary multiplicity in the sum of elementary Prym integrals have been proved.

MATERIALS AND METHODS

1. Preliminaries

Let F be a fixed smooth compact oriented surface of genus $g \geq 2$ with $\{a_k, b_k\}_{k=1}^{g}$, and F_0 be a compact Riemann surface with fixed complex analytic structure on F. Let us fix different points $P_1, ..., P_n \in F$. Let $F' = F \setminus \{P_1, ..., P_n\}$ be a surface of type $(g, n), n \geq 1, g \geq 2$. Any other structure on F' is given by some Bertrami differential μ on F_0', i.e. by the expression of the form $\mu(z)\frac{d\bar{z}}{dz}$, which is invariant relative to the choice of the local parameter on F_0', where $\mu(z)$ is complex-valued function on F_0' and $|\mu|_{L_\infty(F_0')} < 1$. This structure on F' we will denote by F_μ'. It is uniformized by quasi-Fuchsian group Γ_μ'.

In the work of L. Bers [3] Abelian differentials $\zeta_1[\mu], ..., \zeta_g[\mu]$ on F_μ, which form a canonical basis dual to the canonical homotopy basis $\{a_k^\mu, b_k^\mu\}_{k=1}^{g}$ on F_μ have been constructed and also it holomorphically depends on points $[\mu]$ of the Teichmuller space T_g. Moreover, the matrix of $b-$periods $\Omega(\mu) = (\pi_{jk}[\mu])_{j,k=1}^{g}$ on F_μ consists of complex numbers $(\pi_{jk}[\mu]) = \int_{\xi}^{B_k^\mu(\xi)} \zeta_j([\mu], w)dw, \xi \in w^\mu(U)$, and holomorphically depends on $[\mu]$.

For any fixed $[\mu] \in T_g$ and $\xi_0 \in w^\mu(U)$ let us define a classical Jacobi mapping $\varphi : w^\mu(U) \to C^g$

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according to the rule $\varphi_j(\xi)=\int_{\xi_0}^{\xi}\zeta_j([\mu],w)dw$, $j=1,\dots,g$.

A quotient space $J(F)=C^g/L(F)$ is called marked Jacobi manifold for $F=F_0$, where $L(F)$ is a lattice above Z, generated by columns $e^{(1)},\dots,e^{(g)},\pi^{(1)},\dots,\pi^{(g)}$ of the matrix (I_g,Ω) [6, 7].

Any homomorphism $\rho:(\pi_1(F_\mu^{'}),.)\to(C^*,.),C^*=$ $=C\setminus\{0\}$ is called the character ρ for $F_\mu^{'}$. Further we will assume that $\rho(\gamma_j^\mu)=1$, where γ_j^μ is a simple loop that avoid puncture P_j on $F_\mu^{'}, j=1,\dots,n.$

Definition 1.1. Meromorphic function f on $w^\mu(U)$ is called a multiplicative function f on $F_\mu^{'}$ for the character ρ if $f(Tz)=\rho(T)f(z)$, $z\in w^\mu(U),T\in\Gamma_\mu^{'}$.

Definition 1.2. Differential $\omega(z)dz^q$ is called $q-$Prym differential in relation to Fuchsian group $\Gamma_\mu^{'}$ for ρ, or (ρ,q) – differential, if $\omega(Tz)(T'z)^q=$ $=\rho(T)\omega(z),\ z\in U,\ T\in\Gamma',\rho:\Gamma'\to C^*$

If f_0 is a multiplicative function on F_μ for ρ without zeros or poles, then ρ characters for such functions we shall call inessential and f_0 we shall call a unit. Characters that are not inessential we shall call essential on $\pi_1(F^\mu)$. A set L_g of inessential characters forms a subgroup in the group $Hom(\Gamma,C^*)$ of all characters on Γ.

Theorem (of Abel for characters) [2, 4]. Let $D=P_1^{n_1}\dots P_k^{n_k},P_j\in F_\mu,n_j\in Z,j=1,\dots,k,$ be a divisor on a marked variable compact Riemann surface $[F_\mu,a_1^\mu,\dots,a_g^\mu,b_1^\mu,\dots,b_g^\mu]$ of genus $g\ge1$, and let ρ be a character on $\pi_1(F_\mu)$. Then D will be a divisor of the function f on F_μ for $\rho\Leftrightarrow\deg D=0$ and $\varphi(D)=\frac{1}{2\pi i}\sum_{j=1}^{g}\log\rho(b_j^\mu)e^{(j)}[\mu]\quad-\log\rho(a_j^\mu)\pi^{(j)}[\mu]\equiv$ $\equiv\psi(\rho,[\mu])$, where $\varphi[\mu]:F_\mu\to J(F_\mu)$ is a Jacobi mapping.

Class $M_1(\rho)$ consists of Prym differentials for ρ on $F_\mu^{'}$ which have a finite number of poles on $F_\mu^{'}$ and allow meromorphic continuation on F_μ.

In [6, 7] it was proved that for any essential character ρ, points $Q_1\in F_\mu$, of natural number $q\ge1$ and inessential character ρ, points $Q_1\in F_\mu$, of natural number $q>1$ exists elementary $(\rho,q)-$ differential $\tau_{\rho,q;Q_1}$ of third kind with a single simple pole $Q_1[\mu]$ on F_μ. For any inessential character ρ, points $Q_1\in F_\mu$, when $q=1$, elementary $(\rho,1)-$differential $\tau_{\rho,q;Q_1}$ doesn't exist. There it was also proved that on a variable surface F_μ of kind $g\ge2$ for any natural number $q\ge1$ exists elementary $(\rho,q)-$differential $\tau_{\rho,q;Q_1,Q_2}$ of third kind with simple poles $Q_1,Q_2\in F_\mu$, and $\tau^{(m)}_{\rho,q;Q_1}=(\frac{1}{z^m}+O(1))dz^q$ of second genus with the pole $Q_1[\mu]$ of order $m\ge2$. These differentials locally holomorphically depend on $[\mu]$ and ρ.

2. An analogue of the residue theorem for Prym differentials on finite Riemann surface

Residues for Prym differentials can be defined only for the branches of these multivalued differentials.

Let $\tau-$be a $(\rho,q)-$ differential such that $(\tau)\ge\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}},\quad s\ge1,\alpha_j\ge0,\alpha_j\in N,j=1,\dots,s,$ with pairwise distinct points $Q_1,\dots,Q_s$ on F'.

Analytic continuation τ (hereinafter referred to with this symbol) with F' on F, meets conditions

$$(\tau)\ge\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}P_1^{k_1}\dots P_n^{k_n}},\ k_j\ge0,k_j\in N,j=1,\dots,n.$$

Let us introduce the following notations:

1) if ρ is inessential character, then let us choose multiplicative unit on F for ρ^{-1}, where

$$f_0=\exp[-\int_{Q_0}^{P}\sum_{j=1}^{g}\log\rho(a_j)\zeta_j],$$
$$Q_0\ne Q_1,\dots,Q_s,P_1,\dots,P_n;$$

2) if ρ is an essential character, then there exists a single function f_1 on F for ρ^{-1} with a single simple pole P_1 [6]. Such function f_1 has a divisor $\frac{R_1}{P_1}$, where $\varphi(R_1)=\varphi(P_1)-\psi(\rho)$ in Jacobi manifold $J(F),\psi(\rho)\ne0$. This function may be presented in a form of

$$f_1(P)=\exp[\int_{Q_0}^{P}(\tau_{R_1P_1}-\sum_{j=1}^{g}\log\rho(a_j)\zeta_j)],$$
$$Q_0\ne Q_1,\dots,Q_s,P_1,\dots,P_n\ [4].$$

Let us show uniqueness of such function. If there is a point R_2 such that the equality $\varphi(R_1)=\varphi(P_1)-\psi(\rho)=\varphi(R_2)$ is true, then $\varphi(\frac{R_1}{R_2})=0$. According to classical Abel's theorem there is a single-valued function h with the divisor $(h)=\frac{R_1}{R_2}$ that has a single simple pole on compact Riemann surface of a positive genus. A contradiction.

Without loss of generality we can find Abelian holomorphic 1-differential ω_0 such that $(\omega_0)\cap\{Q_1,\dots,Q_s,P_1,\dots,P_n\}=\varnothing$ on F, because divisors of Abelian holomorphic differentials don't have base points on F [4]. Let us choose any Abelian differential ω_0 with the divisor $(\omega_0) = S_1,...,S_{2g-2}$ on F, so that there were as few points as possible in its divisor. We have an equality $\varphi_{P_0}(S_1...S_{2g-2}) = -2K$ in Jacobi manifold $J(F)$, where K is a vector of Riemann constant for a marked compact Riemann surface F with the base point P_0 [4]. This equality is equivalent to another equality in the form $\varphi_{S_{g+1}}(S_1...S_g) = -2K - \varphi_{S_{g+1}}(S_{g+1}^{g-2})$, where $P_0 = S_{g+1} = ... = S_{2g-2}$. This implies Abelian differential ω_0 with divisor in the form $(\omega_0) = S_1...S_g S_{g+1}^{g-2}$.

Let us make Abelian 1-differentials $\frac{\tau f_0}{\omega_0^{q-1}}$ and $\frac{\tau f_1}{\omega_0^{q-1}}$ on F, where f_0 and f_1 have character ρ^{-1} on F and differential τ is analytically continued from F' on F. By the theorem on a complete sum of residues for Abelian 1-differentials $\frac{\tau f_j}{\omega_0^{q-1}}, j = 0,1,$ on F we obtain the following analogue of the theorem on a complete sum of residues for $(\rho, q)-$differentials.

Theorem 2.1.

1) For any $(\rho, q)-$differential τ of the class M_1 on Riemann surface F of type $(g,n), g \geq 2, n \geq 1,$ with any polar divisor $(\tau)_\infty = Q_1^{\alpha_1}...Q_s^{\alpha_s}$, of any integer q and unit f_0 for inessential character ρ^{-1} on F the following equality is true:

$$\sum_{j=1}^{s} res_{Q_j} \frac{\tau f_0}{\omega_0^{q-1}} + \sum_{j=1}^{g+1} res_{S_j} \frac{\tau f_0}{\omega_0^{q-1}} + \sum_{j=1}^{n} res_{P_j} \frac{\tau f_0}{\omega_0^{q-1}} = 0;$$

2) For any $(\rho, q)-$ differential τ of the class M_1 on Riemann surface F of type $(g,n), g \geq 2, n \geq 1,$ with any polar divisor $(\tau)_\infty = Q_1^{\alpha_1}...Q_s^{\alpha_s}$, of any integer q and single, accurate to multiplication by a nonzero constant, function f_1 for essential character $\rho^{-1}, (f_1) \geq \frac{1}{P_1}$ on F the following equality is true:

$$\sum_{j=1}^{s} res_{Q_j} \frac{\tau f_1}{\omega_0^{q-1}} + \sum_{j=1}^{g+1} res_{S_j} \frac{\tau f_1}{\omega_0^{q-1}} + \sum_{j=1}^{n} res_{P_j} \frac{\tau f_1}{\omega_0^{q-1}} = 0.$$

In both cases ω_0 is an Abelian holomorphic differential with a divisor

$$(\omega_0) = S_1...S_g S_{g+1}^{g-2}, \varphi_{S_{g+1}}(S_1...S_g) = -2K$$

in Jacobi manifold $J(F)$ and $\{S_1,...,S_g, S_{g+1}\} \cap \{Q_1,...,Q_s, P_1,...,P_n\} = \varnothing$ on F.

Remark 2.1. Note that in the preceding theorem when $q \leq 0$ there is no second sum in the assertion of the theorem.

Remark 2.2. P. Appell considered the residue theorem only when $q = 0$ on compact Riemann surface of kind $g > 1$.

Let us find some corollaries of the residue theorem and reciprocity laws for multiplicative functions on finite Riemann surface. First we find corollary for 1-differential ω with any character ρ, in a special case, when $\omega = df$, where f is a multiplicative function on F' of the M_1 class.

Let $R_1,...,R_m$ be zeros of f with multiplicity $\lambda_1,....,\lambda_m$ and let $Q_1,...,Q_s$ be poles for f with multiplicity $\mu_1,....,\mu_s$ when the function f is continued analytically from F' to F. Let us also consider single-valued function h on F with poles $L_1,...,L_l$ of multiplicity $p_1,...,p_l$ accordingly where points L_i are not included in the support of the divisor $\sup p(f)$. Let us note that $\frac{df}{f}$ will be an Abelian differential with simple poles $R_1,...,R_m, Q_1,...,Q_s$ and residues $\lambda_1,...,\lambda_m, -\mu_1,...,-\mu_s$ in them accordingly. Then, because of uniqueness of the expression under integral and by the residue theorem for Abelian 1-differentials, we obtain that:

$$0 = \frac{1}{2\pi i} \int_{\partial\Delta_\mu} h \frac{df}{f} = \sum_{k=1}^{m} \lambda_k h(R_k) - \sum_{j=1}^{s} \mu_k h(Q_j) + \sum_{i=1}^{l} res_{L_i} h \frac{df}{f}, \quad (1)$$

where Δ is a connected fundamental domain for group Γ in domain U [3, 5].

From (1) we will obtain formulas that are connected with a special choice of a function h on F:

1) Let h be an analytical function on F, and h be a constant. Then $\sum_{k=1}^{m} \lambda_k = \sum_{j=1}^{s} \mu_j$ is a classical fact that $\deg(f) = 0$ on F [4];

2) If h has multiple poles in points L_i, i.e.

$$h = \frac{c_{-p_i}^{(i)}}{(z - z(L_i))^{p_i}} + ... + \frac{c_{-1}^{(i)}}{z - z(L_i)} + O(1), i = 1,...,l,$$

then we obtain the equality

$$\sum_{j=1}^{s} \mu_j h(Q_j) - \sum_{k=1}^{m} \lambda_k h(R_k) = \\ = \sum_{i=1}^{l} \left[(\frac{f'}{f})^{(p_i-1)}(L_i) \frac{c_{-p_i}^{(i)}}{(p_i-1)!} + (\frac{f'}{f})^{(p_i-2)}(L_i) \frac{c_{-p_i+1}^{(i)}}{(p_i-2)!} + ... + (\frac{f'}{f})'(L_i) c_{-2}^{(i)} + (\frac{f'}{f})(L_i) c_{-1}^{(i)} \right].$$

Remark 2.3. These two equalities are some reciprocity laws that connect zeros and poles for the multiplicative function f of the class M_1 on F' with poles of single-valued meromorphic functions h on F.

3. An analogue of Appell's formula for a multiplicative function expansion on a variable finite Riemann surface.

Let us denote through $T^{(1)}_{\rho;Q}(z) = -\int \tau^{(2)}_{\rho;Q}$ an elementary Prym integral of the second kind on F_μ for essential character ρ with a single simple pole in Q and residue +1 in Q which holomorphically depends on $[\mu]$ and ρ, where $\tau^{(2)}_{\rho;Q}$ has a zero residue in the point Q.

Let f be a function on F'_μ of the class M_1 for essential character ρ with s simple poles $P_{n+1}, P_{n+2}, ..., P_{n+s}$ and residues $c_{n+1}, ..., c_{n+s}$ in them accordingly for one of its branches. Let us take analytical continuation of this function (and denote it with the same symbol) f with F'_μ on F'_μ. Let us consider the expression

$$f_1 = f - c_{n+1}T^{(1)}_{\rho;P_{n+1}} - ... - c_{n+s}T^{(1)}_{\rho;P_{n+s}} - \sum_{j=1}^{g-1} \tilde{c}_j \int \tilde{\zeta}_j,$$

where $\tilde{c}_j \in C, j = 1, ..., g-1,$ and $\tilde{\zeta}_1, ..., \tilde{\zeta}_{g-1}$ are the basis of Prym differentials of the first kind for essential character ρ on F_μ, that holomorphically depends on $[\mu]$ and ρ [2]. Then f_1 is a meromorphic single-valued branch of the Prym integral with essential character ρ on fundamental polygon Δ_μ with divisor

$$(f_1) \geq \frac{1}{P_1^{k_1}...P_n^{k_n}}, \quad k_j \geq 0, j = 1, ..., n, \text{ on } F_\mu.$$

Among other things, Prym integral f_1 for ρ has a branch whose principal parts of Laurent series match with principal parts of Laurent series in points $P_j, j = 1, ..., n,$ for f and zeros a_m are periods, $m = 1, ..., g-1,$ on F_μ [2]. Therefore

$$f = \sum_{j=1}^{s} c_{n+j} T^{(1)}_{\rho;P_{n+j}} + \sum_{j=1}^{g-1} \tilde{c}_j \int \tilde{\zeta}_j + f_1.$$

If P_{n+1} is a pole of $k_{n+1}, k_{n+1} \geq 2$ order, then in the preceding formula a summand $c_{n+1}\, T^{(1)}_{\rho;P_{n+1}}$ should be replaced with the sum of the form

$$A_{n+1,1}\, T^{(1)}_{\rho;P_{n+1}} + A_{n+1,2} \frac{\partial T^{(1)}_{\rho;P_{n+1}}}{\partial P_{n+1}} +$$

$$+ \frac{A_{n+1,3}}{2} \frac{\partial^2 T^{(1)}_{\rho;P_{n+1}}}{\partial P^2_{n+1}} + ... + \frac{A_{n+1,k_{n+1}}}{(k_{n+1}-1)!} \frac{\partial^{k_{n+1}-1} T^{(1)}_{\rho;P_{n+1}}}{\partial P^{k_{n+1}-1}_{n+1}},$$

where $A_{n+1,1}$ are coefficients in the principal part of Laurent series for some branch of the function f in the point $P_{n+1}, j = 1, ..., k_{n+1}(P_{n+1})$. Indeed, in the neighborhood of the point P_{n+k} we have expansion

$$T^{(1)}_{\rho;P_{n+k}} = \frac{1}{(z - z(P_{n+k}))} + O(1); (T^{(1)}_{\rho;P_{n+k}})'_{a_k} =$$

$$= \frac{1}{(z-a_k)^2} + O(1), z(P_{n+k}) = a_k; ...$$

$$(T^{(1)}_{\rho;P_{n+k}})^{(m)}_{a_k} = \frac{m!}{(z-a_k)^{m+1}} + O(1), 1 \leq m \leq k_{n+k}(P_{n+k}) - 1,$$

where $k_{n+k}(P_{n+k})$ is an order of a pole in the point P_{n+k} for f , k = 1,…,s.

It follows that

Theorem 3.1. Let f be a branch of the function of the class M_1 for essential character ρ on a variable Riemann surface F'_μ of type $(g,n), g \geq 2, n > 0,$ with pairwise different poles in $P_{n+1}, ..., P_{n+s}$ of the multiplicities $k_{n+1}, ..., k_{n+s}$ with given principal parts in them.

Then for the analytical continuation f it is true that

$$(f) \geq \frac{1}{P_1^{k_1}...P_{n+s}^{k_{n+s}}}, k_j \geq 0, j = 1, ..., n+s,$$

on F_μ and

$$f = \sum_{j=1}^{n+s} \sum_{m=1}^{k_j} \left[\frac{A_{j,m}}{(m-1)!} \frac{\partial^{m-1} T^{(1)}_{\rho;P_j}}{\partial P_j^{m-1}} \right] + \sum_{j=1}^{g-1} \tilde{c}_j \int_{P_0}^{P} \tilde{\zeta}_j,$$

where

$$f = \frac{A_{j,k_j}}{(z - z(P_j))^{k_j}} + ... + \frac{A_{j,2}}{(z-z(P_j))^2} + \frac{A_{j,1}}{(z - z(P_j))} + O(1) \text{ f}$$

or some branch in the neighborhood $P_j, j = 1, ..., n+s$ on F_μ, and all summands holomorphically depend on $[\mu]$ and ρ.

Now let ρ be an inessential character. The proof of the preceding expansion formula for essential character is not applicable because in this case Prym integral of the second kind with a single simple pole on F_μ doesn't exist. That is Prym differential of the second kind for inessential character ρ has to have at least two second-order poles in different arbitrary points Q_1 and Q_2 on Δ_μ, and with zero residues in Q_1 and Q_2. In this case Prym integrals $T_{\rho;Q_1,Q_2} = -\int_{Q_0}^{P} \tau_{\rho;Q_1^2 Q_2^2}$ of the second kind with two simple poles Q_1 and Q_2 should be used as prime elements of expansion.

Let us consider another Prym differential $\tau_{\rho;Q_1 Q_2} = f_0 \tau_{Q_1 Q_2}$ of the third kind on F_μ, where f_0 is a unit for ρ on F_μ and $\tau_{Q_1 Q_2}$ is a normalized Abelian differential with simple poles Q_1 and Q_2 on F_μ, and residues +1 and -1 in these points accordingly, which holomorphically depend on $[\mu]$ and ρ. It has been known that $\tau_{Q_1 Q_2} = d\Pi_{Q_1 Q_2}$ and Abelian differential $\Pi_{Q_1 Q_2}$ is expressed through the Riemann theta function for the surface F_μ. In such case it is equal to the sum of two functions, one of which only depends on Q_1, and another only depends on Q_2 [2, p. 117]. Thus a derivative $\frac{\partial \Pi_{Q_1 Q_2}}{\partial Q_1}$ doesn't depend on Q_2.

Prym differential $\tau^{(2)}_{\rho;Q_1}$ has expansion $(\frac{1}{(z-z_1)^2}+\frac{c^{(1)}_{-1}}{(z-z_1)}+O(1))dz$ in the neighborhood of the point $Q_1, z(Q_1)=z_1$, where $c^{(1)}_{-1}=\sum_{j=1}^{g}\log\rho(a_j)\varphi'_j(Q_1)$ [6].

Prym differential $\tau^{(2)}_{\rho;Q_2}$ also has expansion $(\frac{1}{(z-z_2)^2}+\frac{c^{(2)}_{-1}}{(z-z_2)}+O(1))dz$ in the neighborhood of the point $Q_2, z(Q_2)=z_2$, where $c^{(2)}_{-1}=\sum_{j=1}^{g}\log\rho(a_j)\varphi'_j(Q_2)$.

A differential with two poles of the second order and zero residues in these points can be set in the following way

$$\tau_{\rho;Q_1^2 Q_2^2}=c^{(2)}_{-1}f_0(Q_1)\tau^{(2)}_{\rho;Q_1}-c^{(1)}_{-1}f_0(Q_2)\tau^{(2)}_{\rho;Q_2}-$$
$$-c^{(1)}_{-1}c^{(2)}_{-1}\tau_{\rho;Q_1Q_2}.$$

Let us denote that the principal part for $\tau_{\rho;Q_1Q_2}$ in the point Q_1 takes the form of $\frac{f_0(Q_1)}{z-z_1}$ and in the point Q_2 takes the form of $\frac{f_0(Q_2)}{z-z_2}$. It follows that the constructed differential $\tau_{\rho;Q_1^2Q_2^2}$ has two poles of the second order in Q_1 and Q_2, and two residues in this points. In the neighborhood of the point Q_1 its principal part takes the form of

$$c^{(2)}_{-1}f_0(Q_1)[\frac{1}{(z-z_1)^2}+\frac{c^{(1)}_{-1}}{z-z_1}]-$$
$$-c^{(1)}_{-1}c^{(2)}_{-1}\frac{f_0(Q_1)}{z-z_1}=\frac{c^{(2)}_{-1}f_0(Q_1)}{(z-z_1)^2};$$

that is similar in the point Q_2

$$c^{(1)}_{-1}\ f_0(Q_2)[\frac{1}{(z-z_2)^2}+\frac{c^{(2)}_{-1}}{z-z_2}]+$$
$$+c^{(1)}_{-1}c^{(2)}_{-1}\frac{f_0(Q_2)}{z-z_2}=-\frac{c^{(1)}_{-1}f_0(Q_2)}{(z-z_2)^2}.$$

It is clear that a constructed differential $\tau_{\rho;Q_1^2Q_2^2}$ holomorphically depends on $[\mu]$ and ρ.

From all has been said it follows that derivative $\frac{\partial T_{\rho;Q_1Q_2}}{\partial Q_1}$ doesn't depend on Q_2.

Theorem 3.2. Let f be a branch of the function of the class M_1 with inessential character ρ and pairwise different poles $P_{n+1},\ldots,P_{n+s}$ of the multiplicity $k_{n+1},\ldots,k_{n+s}$ with set principal parts in them on a variable Riemann surface F'_μ of type $(g,n), g\geq 2, n>0$. Then for the analytical continution f on F_μ it is true that

$$(f)\geq\frac{1}{P_1^{k_1}\ldots P_{n+s}^{k_{n+s}}}, k_j\geq 0, j=1,\ldots,n+s,$$

and

$$f(P)=\sum_{j=1}^{g}c_j\int_{P_0}^{P}f_0\zeta_j+\sum_{r=1}^{n+s-1}\frac{A_{r1}T_{\rho;P_{n+s}P_r}}{d_{n+s}f_0(P_r)}+$$
$$+\sum_{m=2}^{k_1}\frac{A_{1m}}{(m-1)!}\frac{\partial^{m-1}T_{\rho;P_1P_{n+1}}}{\partial P_1^{m-1}}+$$
$$+\sum_{j=2}^{n+s}[A_{j,2}\frac{\partial T_{\rho;P_jP_1}}{\partial P_j}+\frac{A_{j,3}}{2!}\frac{\partial^2 T_{\rho;P_jP_1}}{\partial P_j^2}+\ldots+\frac{A_{j,k_j}}{(k_j-1)!}\frac{\partial^{k_j-1}T_{\rho;P_jP_1}}{\partial P_j^{k_j-1}}]+C,$$

where

$$f=\frac{A_{j,k_j}}{(z-z(P_j))^{k_j}}+\ldots+\frac{A_{j,2}}{(z-z(P_j))^2}+\frac{A_{j,1}}{z-z(P_j)}+O(1)$$

for some branch in the neighborhood $P_j, j=1,\ldots,n+s$, on F_μ, $C=0$ when $\rho\geq 1$,

$$d_k=\sum_{m=1}^{g}\log\rho(a_m)\varphi'_m(P_k), k=1,\ldots,n+s,$$

on F_μ, and all summands holomorphically depend on $[\mu]$ and ρ.

Proof. It suffices to verify a coincidence of principal parts on the left and right in this formula. For the neighborhood of the point $P_r, r=1,\ldots,n+s$, on Δ_μ we have expansion in Laurent series

$$\sum_{r=1}^{m+s-1}(\frac{d_{n+s}f_0(P_r)}{z-P_r}-\frac{d_rf_0(P_{n+s})}{z-P_{n+s}})\frac{A_{r1}}{d_{n+s}f_0(P_r)}=$$
$$=\frac{A_{r1}}{z-P_r}+\ldots, r=1,\ldots,n+s-1.$$

For the neighborhood of the point P_{n+s} we have expansion

$$\sum_{r=1}^{n+s-1}\frac{-d_rf_0(P_{n+s})}{z-P_{n+s}}\frac{A_{r1}}{d_{n+s}f_0(P_r)}=$$
$$=\frac{1}{z-P_{n+s}}\frac{f_0(P_{n+s})}{d_{n+s}}\sum_{r=1}^{n+s-1}\frac{-d_rA_{r1}}{f_0(P_r)}+\ldots=\frac{A_{n+s,1}}{z-P_{n+s}}+\ldots,$$

because

$$\sum_{r=1}^{n+s}\frac{-A_{r1}d_r}{f_0(P_r)}=0,\ \frac{f_0(P_{n+s})}{d_{n+s}}\sum_{r=1}^{n+s-1}\frac{-d_rA_{r1}}{f_0(P_r)}=A_{n+s,1},$$

according to the formula on full sum of residues for Abelian differentials $\frac{f}{f_0}d(\sum_{j=1}^{g}\log\rho(a_j)\varphi_j)$ of third kind on F_μ, which in the point P_j has residue $\frac{A_{j1}d_j}{f_0(P_j)}, j=1,\ldots,n+s$. The theorem is proved.

Remark 3.1. P. Appell [2] has proven the theorem 3.2 for compact Riemann surface and simple poles.

Every simple element (summand) depended on additional $g-1$ poles. In our work the theorem has been proved for a variable finite Riemann surface F'_μ of genus $(g,n), g \geq 1, n > 0,$ and poles of any multiplicity. Moreover, every summand in our work has either one or two poles. Also when $\rho = 1$, $n = 0$ we obtain a classical fact about expansion of single-valued meromorphic function in sum of Abelian integrals on a compact Riemann surface.

CONCLUSION

Analogues of the residue theorem for Prym differentials of any entire order on variable finite Riemann surfaces are obtained for the first time. Thus three reciprocity laws have been proved. Analogues of P.Appell's expansion formula for functions with any characters on variable finite Riemann surfaces have been proved. In this case simple elements (summands) only have one or two poles.

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